2021

AP CollegeBoard

AP® Calculus BC

Sample Student Responses and Scoring Commentary

Inside:

Free Response Question 1

- ☑ Scoring Guideline
- ☑ Student Samples
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Part A (AB or BC): Graphing calculator required

Question 1 — 9 points

General Scoring Notes

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.

r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

The density of a bacteria population in a circular petri dish at a distance r centimeters from the center of the dish is given by an increasing, differentiable function f, where $f(r)$ is measured in milligrams per square centimeter. Values of $f(r)$ for selected values of r are given in the table above.

Model Solution	Scoring	
(a) Use the data in the table to estimate $f'(2.25)$. Using correct units, interpret the meaning of your answer in the context of this problem.		
$f'(2.25) \approx \dfrac{f(2.5)-f(2)}{2.5-2} = \dfrac{10-6}{0.5} = 8$	Estimate	**1 point**
At a distance of $r = 2.25$ centimeters from the center of the petri dish, the density of the bacteria population is increasing at a rate of 8 milligrams per square centimeter per centimeter.	Interpretation with units	**1 point**

Scoring notes:

- To earn the first point the response must provide both a difference and a quotient and must explicitly use values of f from the table.
- Simplification of the numerical value is not required to earn the first point. If the numerical value is simplified, it must be correct.
- The interpretation requires all of the following: distance $r = 2.25,$ density of bacteria (population) is increasing or changing, at a rate of $8,$ and units of milligrams per square centimeter per centimeter.
- The second point (interpretation) cannot be earned without a nonzero presented value for $f'(2.25).$
- To earn the second point the interpretation must be consistent with the presented nonzero value for $f'(2.25).$ In particular, if the presented value for $f'(2.25)$ is negative, the interpretation must include "decreasing at a rate of $|f'(2.25)|$" or "changing at a rate of $f'(2.25).$" The second point cannot be earned for an incorrect statement such as "the bacteria density is decreasing at a rate of -8 … " even for a presented $f'(2.25) = -8.$
- The units ($\text{mg/cm}^2\text{/cm}$) may be attached to the estimate of $f'(2.25)$ and, if so, do not need to be repeated in the interpretation.
- If units attached to the estimate do not agree with units in the interpretation, read the units in the interpretation.

Total for part (a) 2 points

(b) The total mass, in milligrams, of bacteria in the petri dish is given by the integral expression $2\pi\int_0^4 r\,f(r)\,dr$. Approximate the value of $2\pi\int_0^4 r\,f(r)\,dr$ using a right Riemann sum with the four subintervals indicated by the data in the table.

$2\pi\int_0^4 r\,f(r)\,dr \approx 2\pi(1\cdot f(1)\cdot(1-0) + 2\cdot f(2)\cdot(2-1) + 2.5\cdot f(2.5)\cdot(2.5-2) + 4\cdot f(4)\cdot(4-2.5))$	Right Riemann sum setup	**1 point**
$= 2\pi(1\cdot 2\cdot 1 + 2\cdot 6\cdot 1 + 2.5\cdot 10\cdot 0.5 + 4\cdot 18\cdot 1.5)$ $= 269\pi = 845.088$	Approximation	**1 point**

Scoring notes:

- The presence or absence of 2π has no bearing on earning the first point.
- The first point is earned for a sum of four products with at most one error in any single value among the four products. Multiplication by 1 in any term does not need to be shown, but all other products must be explicitly shown.
- A response of $1\cdot f(1)\cdot(1-0) + 2\cdot f(2)\cdot(2-1) + 2.5\cdot f(2.5)\cdot(2.5-2) + 4\cdot f(4)\cdot(4-2.5)$ earns the first point but not the second.
- A response with any error in the Riemann sum is not eligible for the second point.
- A response that provides a completely correct left Riemann sum for $2\pi\int_0^4 r\,f(r)\,dr$ and approximation (91π) earns one of the two points. A response that has any error in a left Riemann sum or evaluation for $2\pi\int_0^4 r\,f(r)\,dr$ earns no points.
- A response that provides a completely correct right Riemann sum for $2\pi\int_0^4 f(r)\,dr$ and approximation (80π) earns one of the two points. A response that has any error in a right Riemann sum or evaluation for $2\pi\int_0^4 f(r)\,dr$ earns no points.
- Simplification of the numerical value is not required to earn the second point. If a numerical value is given, it must be correct to three decimal places.

Total for part (b) 2 points

(c) Is the approximation found in part (b) an overestimate or underestimate of the total mass of bacteria in the petri dish? Explain your reasoning.

$\frac{d}{dr}(r\,f(r)) = f(r) + r\,f'(r)$	Product rule expression for $\frac{d}{dr}(r\,f(r))$	**1 point**
Because f is nonnegative and increasing, $\frac{d}{dr}(r\,f(r)) > 0$ on the interval $0 \le r \le 4.$ Thus, the integrand $r\,f(r)$ is strictly increasing. Therefore, the right Riemann sum approximation of $2\pi\int_0^4 r\,f(r)\,dr$ is an overestimate.	Answer with explanation	**1 point**

Scoring notes:

- To earn the second point a response must explain that $r\,f(r)$ is increasing and, therefore, the right Riemann sum is an overestimate. The second point can be earned without having earned the first point.
- A response that attempts to explain based on a left Riemann sum for $2\pi\int_0^4 r\,f(r)\,dr$ from part (b) earns no points.
- A response that attempts to explain based on a right Riemann sum for $2\pi\int_0^4 f(r)\,dr$ from part (b) earns no points.

Total for part (c) 2 points

(d) The density of bacteria in the petri dish, for $1 \le r \le 4$, is modeled by the function g defined by $g(r) = 2 - 16\left(\cos\left(1.57\sqrt{r}\right)\right)^3$. For what value of k, $1 < k < 4$, is $g(k)$ equal to the average value of $g(r)$ on the interval $1 \le r \le 4$?

Average value $= g_{\text{avg}} = \frac{1}{4-1}\int_1^4 g(r)\,dr$	Definite integral	**1 point**
$\frac{1}{4-1}\int_1^4 g(r)\,dr = 9.875795$	Average value	**1 point**
$g(k) = g_{\text{avg}} \Rightarrow k = 2.497$	Answer	**1 point**

Scoring notes:

- The first point is earned for a definite integral, with or without $\frac{1}{4-1}$ or $\frac{1}{3}$.
- A response that presents a definite integral with incorrect limits but a correct integrand earns the first point.
- Presentation of the numerical value 9.875795 is not required to earn the second point. This point can be earned by the average value setup: $\frac{1}{3}\int_1^4 g(r)\,dr$.
- Once earned for the average value setup, the second point cannot be lost. Subsequent errors will result in not earning the third point.
- The third point is earned only for the value $k = 2.497$.
- The third point cannot be earned without the second.
- Special case: A response that does not provide the average value setup but presents an average value of -13.955 is using degree mode on their calculator. This response would not earn the second point but could earn the third point for an answer of $k = 2.5$ (or 2.499).

Total for part (d) 3 points

Total for question 1 9 points

Answer QUESTION 1 parts (a) and (b) on this page.

r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

Response for question 1(a)

$f'(2.25) \approx \frac{f(2.5)-f(2)}{2.5-2} = \frac{10-6}{0.5} = \frac{4}{0.5} = 8$ milligrams per square centimeter per centimeter

The density of bacteria changes at a rate of approximately 8 milligrams per square centimeter per centimeter at distance $r = 2.25$ centimeters from the center of the dish

Response for question 1(b)

$2\pi\int_0^4 r\cdot f(r)\,dr \approx 2\pi(2\cdot1\cdot1 + 6\cdot1\cdot2 + 10\cdot0.5\cdot2.5 + 18\cdot1.5\cdot4)$

$= 2\pi(2+12+12.5+108) = 2\pi(134.5) = 269\pi$ milligrams

Answer QUESTION 1 parts (c) and (d) on this page.

Response for question 1(c)

As a rule, Right riemann Sums are always an over estimate for functions with possitive slope, and underestimates for functions with negative slope. The slope of $r \cdot f(r)$ is equal to $r' \cdot f(r) + r \cdot f'(r)$. Since r, r', $f(r)$, and $f'(r)$ are always positive on the interval $[0,4]$, $r \cdot f(r)$ always has a positive slope on that interval. Since it's a positive sloped function, the right riemann sum for $r \cdot f(r)$ from 0 to 4 is an over estimate.

Response for question 1(d)

$$\text{Avg of } g(r) = \frac{1}{4-1}\int_1^4 \left(2-16(\cos(1.57\sqrt{r}))^3\right)dr = \frac{29.627}{3} = 9.876$$

$$g(k) = 2-16(\cos(1.57\sqrt{k}))^3 = 9.876$$

$$k = 2.497$$

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Answer QUESTION 1 parts (a) and (b) on this page.

r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

Response for question 1(a)

$$f'(2.25) \approx \frac{f(2.5)-f(2)}{2.5-2} = \frac{10-6}{0.5} = 8 \ \frac{\text{milligrams}}{\text{cm}^3}$$

The rate of change of f(r) at distance r = 2.25 centimeters is approximately 8 milligrams per cubic centimeter

Response for question 1(b)

$$2\pi\int_0^4 r f(r)\,dr \approx 2\pi\left[1(1)f(1) + 1(2)f(2) + 0.5(2.5)f(2.5) + 1.5(4)f(4)\right]$$

$$\approx 2\pi[2 + 12 + 12.5 + 108] = 269\pi$$

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Answer QUESTION 1 parts (c) and (d) on this page.

Response for question 1(c)

for [0,4] f(r) is increasing (as is r)
so r f(r) is increasing.
As a right Riemann sum was used ~~to~~ to take the integral of an increasing function, it was an overestimate

Response for question 1(d)

Avg. value of $g(r)$ on $[1,4] = \frac{1}{4-1}\int_1^4 (2-16(\cos(1.57\sqrt{r}))^3)\,dr = 9.87579487$

$2-16(\cos(1.57\sqrt{K}))^3 = 9.875794868$

at $K = 2.497$ centimeters

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Answer QUESTION 1 parts (a) and (b) on this page.

r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

Response for question 1(a)

$f'(2.25) = \frac{f(2.5)-f(2)}{2.5-2} \rightarrow \frac{10-6}{2.5-2} = \frac{4}{0.5} = 8$

$f'(2.25) = 8$ mg per cm^2 per cm^2

At $r = 2.25$, the rate of the density of the bacteria population is increasing at a rate of 8 mg per cm^2 per cm^2

Response for question 1(b)

$2\pi \cdot [1(2) + 1(6) + 0.5(10) + 1.5(18)]$

$2\pi \cdot (2+6+5+27)$

$2\pi(40) = \boxed{80\pi = 251.327}$

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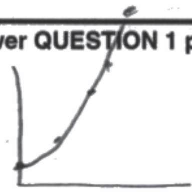
Answer QUESTION 1 parts (c) and (d) on this page.

Response for question 1(c)



The right Riemann sum approximation is an overestimate due to the fact that the total mass of the bacteria is increasing since its represented by f(r) which is the function of the bacteria's density which is an increasing differentiable function

Response for question 1(d)

$g(r) = 2 - 16(\cos(1.57\sqrt{r}))^3$

$\frac{1}{4-1}\int_1^4 g(r)\,dr = 4.875794868$

$g'(K) = 4.875794868$

Question 1

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

The context of this problem is bacteria in a circular petri dish. The increasing, differentiable function f gives the density of the bacteria population (in milligrams per square centimeter) at a distance r centimeters from the center of the dish. Selected values of $f(r)$ are provided in a table.

In part (a) students were asked to use the table to estimate $f'(2.25)$ and interpret the meaning of this value in context, using correct units. A correct response should estimate the derivative value using a difference quotient, drawing from the data in the table that most tightly bounds $r = 2.25.$ The interpretation should explain that when $r = 2.25$ centimeters from the center of the petri dish, the density of the bacteria population is increasing at a rate of roughly 8 milligrams per square centimeter per centimeter.

In part (b) students were told that $2\pi\int_0^4 r\,f(r)\,dr$ gives the total mass, in milligrams, of the bacteria in the petri dish. They were asked to estimate the value of this integral using a right Riemann sum with the values given in a table. A correct response should multiply the sum of the four products $r_i \cdot f(r_i) \cdot \Delta r_i$ drawn from the table by $2\pi.$

In part (c) students were asked to explain whether the right Riemann sum approximation found in part (b) was an overestimate or an underestimate of the total mass of bacteria. A correct response should determine the derivative of $r \cdot f(r)$ using the product rule, use the given information that f is nonnegative to conclude that this derivative is positive and, therefore, that the integrand is strictly increasing on the interval $0 \le r \le 4.$ This means that the right Riemann sum approximation is an overestimate.

In part (d) another function, $g(r) = 2 - 16\left(\cos\left(1.57\sqrt{r}\right)\right)^3$, was introduced as a function that models the density of the bacteria in the petri dish for $1 \le r \le 4.$ Students were asked to find the value of k such that $g(k)$ is equal to the average value of $g(r)$ on the interval $1 \le r \le 4.$ A correct response should set up the average value of $g(r)$ as $\frac{1}{3}\int_1^4 g(r)\,dr,$ then use a graphing calculator to solve for k when setting $g(k)$ equal to this average value.

Sample: 1A
Score: 9

The response earned 9 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and 3 points in part (d). In part (a) the difference quotient of $\frac{10-6}{0.5}$ in the first line would have earned the first point with no simplification. In this case, correct simplification to 8 in the first line earned the first point. The response earned the second point for an interpretation of the density of the bacteria changing at 8 milligrams per square centimeter per centimeter at $r = 2.25.$ In part (b) the response earned the first point for the sum of products expression $2\pi \cdot (2 \cdot 1 \cdot 1 + 6 \cdot 1 \cdot 2 + 10 \cdot 0.5 \cdot 2.5 + 18 \cdot 1.5 \cdot 4)$ in the first line on the right. This sum of products expression would also have earned the second point with no simplification. In this case, correct simplification to 269π in the second line earned the second point. In part (c) the response earned the first point for the product rule expression of $r' \cdot f(r) + r \cdot f'(r)$ for $\frac{d}{dr}(r\,f(r))$ in the fourth line. The response earned the second point for the conclusion that $r\,f(r)$ has a positive slope because $r, r', f(r),$ and $f'(r)$ are positive on the interval and, therefore, the estimate is an overestimate. In part (d) the response earned the first and second points for the definite integral $\frac{1}{4-1}\int_1^4 \left(2 - 16\left(\cos\left(1.57\sqrt{r}\right)\right)^3\right)dr$ giving the average value in the first line. The response earned the third point for the correct value of $k = 2.497$ in the third line.

Question 1 (continued)

Sample: 1B
Score: 7

The response earned 7 points: 1 point in part (a), 2 points in part (b), 1 point in part (c), and 3 points in part (d). In part (a) the difference quotient of $\frac{10-6}{0.5}$ in the first line would have earned the first point with no simplification. In this case, correct simplification to 8 in the first line earned the first point. The response did not earn the second point because the density of the bacteria is not referenced. In part (b) the response earned the first point for the sum of products expression $2\pi[1(1)f(1)+1(2)f(2)+0.5(2.5)f(2.5)+1.5(4)f(4)]$ in the first line. The sum of products expression $2\pi[2+12+12.5+108]$ in the second line would have earned the second point with no simplification. In this case, simplification to 269π in the second line earned the second point. In part (c) the response did not earn the first point because there is no product rule expression for $\frac{d}{dr}(r\,f(r))$. The response earned the second point for the claim that $r\,f(r)$ is increasing in the second line and, therefore, the right Riemann sum is an overestimate in the fifth line. In part (d) the response earned the first and second points for the definite integral $\frac{1}{4-1}\int_1^4\left(2-16\left(\cos\left(1.57\sqrt{r}\right)\right)^3\right)dr$ giving the average value in the first line. The response earned the third point for the correct value of $k = 2.497$ in the third line.

Sample: 1C
Score: 4

The response earned 4 points: 1 point in part (a), 1 point in part (b), no points in part (c), and 2 points in part (d). In part (a) the difference quotient of $\frac{10-6}{2.5-2}$ in the first line would have earned the first point with no simplification. In this case, correct simplification to 8 in the first line earned the first point. The response did not earn the second point because the response states the rate of the density of the bacteria population is increasing at a rate and because the units in the interpretation are incorrect. In part (b) the response earned one of the two points for providing a completely correct right Riemann sum for $2\pi\int_0^4 f(r)\,dr$. The sum of products expression $2\pi\cdot[1(2)+1(6)+0.5(10)+1.5(18)]$ in the first line would have earned one of the two points with no simplification. In this case, correct simplification to 251.327 in the third line earned one of the two points. In part (c) the response did not earn either point because the response attempted to explain a right Riemann sum for $2\pi\int_0^4 f(r)\,dr$. In part (d) the response earned the first and second points for the definite integral $\frac{1}{4-1}\int_1^4 g(r)\,dr$ giving the average value in the second line. The response did not earn the third point because no value is given for k.

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AP® Calculus BC

Sample Student Responses and Scoring Commentary

Inside:

Free Response Question 2

☑ Scoring Guideline

☑ Student Samples

☑ Scoring Commentary

Part A (BC): Graphing calculator required

Question 2 — 9 points

General Scoring Notes

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.

For time $t \geq 0,$ a particle moves in the xy-plane with position $(x(t), y(t))$ and velocity vector $\left\langle (t-1)e^{t^2}, \sin\left(t^{1.25}\right)\right\rangle$. At time $t = 0,$ the position of the particle is $(-2, 5)$.

	Model Solution	Scoring	
(a)	Find the speed of the particle at time $t = 1.2$. Find the acceleration vector of the particle at time $t = 1.2$.		
	$\sqrt{(x'(1.2))^2 + (y'(1.2))^2} = 1.271488$ At time $t = 1.2,$ the speed of the particle is 1.271.	Speed	**1 point**
	$\langle x''(1.2), y''(1.2)\rangle = \langle 6.246630, 0.405125\rangle$ At time $t = 1.2,$ the acceleration vector of the particle is $\langle 6.247 \text{ (or } 6.246), 0.405\rangle$.	Acceleration vector	**1 point**

Scoring notes:

- Unsupported answers do not earn any points in this part.
- The acceleration vector may be presented with other symbols, for example $(\, , \,)$ or $[\, , \,]$, or the coordinates may be listed separately, as long as they are labeled.
- Degree mode: A response that presents answers obtained by using a calculator in degree mode does not earn the first point it would have otherwise earned. The response is generally eligible for all subsequent points (unless no answer is possible in degree mode or the question is made simpler by using degree mode). In degree mode, speed $= 0.844$ and $y''(1.2) = 0.023$ (or 0.022).

Total for part (a) — 2 points

(b) Find the total distance traveled by the particle over the time interval $0 \le t \le 1.2$.

$\int_0^{1.2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = 1.009817$	Integrand	**1 point**
The total distance traveled by the particle over the time interval $0 \le t \le 1.2$ is 1.010 (or 1.009).	Answer	**1 point**

Scoring notes:

- The first point is earned by presenting the integrand $\sqrt{(x'(t))^2 + (y'(t))^2}$ in a definite integral with any limits. A definite integral with incorrect limits is not eligible for the second point.
- Once earned, the first point cannot be lost. Even in the presence of subsequent copy errors, the correct answer will earn the second point.
- If the first point is not earned because of a copy error, the second point is still earned for a correct answer.
- Unsupported answers will not earn either point.
- Degree mode: distance $= 0.677$ (or 0.676). (See degree mode statement in part (a).)

Total for part (b) 2 points

(c) Find the coordinates of the point at which the particle is farthest to the left for $t \geq 0$. Explain why there is no point at which the particle is farthest to the right for $t \geq 0$.

$x'(t) = (t-1)e^{t^2} = 0 \Rightarrow t = 1$	Sets $x'(t) = 0$	**1 point**
Because $x'(t) < 0$ for $0 < t < 1$ and $x'(t) > 0$ for $t > 1$, the particle is farthest to the left at time $t = 1$.	Explains leftmost position at $t = 1$	**1 point**
$x(1) = -2 + \int_0^1 x'(t)\,dt = -2.603511$ $y(1) = 5 + \int_0^1 y'(t)\,dt = 5.410486$	One coordinate of leftmost position	**1 point**
The particle is farthest to the left at point $(-2.604 \text{ (or } -2.603), 5.410)$.	Leftmost position	**1 point**
Because $x'(t) > 0$ for $t > 1$, the particle moves to the right for $t > 1$. Also, $x(2) = -2 + \int_0^2 x'(t)\,dt > -2 = x(0)$, so the particle's motion extends to the right of its initial position after time $t = 1$. Therefore, there is no point at which the particle is farthest to the right.	Explanation	**1 point**

Scoring notes:

- The second point is earned for presenting a valid reason why the particle is at its leftmost position at time $t = 1$. For example, a response could present the argument shown in the model solution, or it could indicate that the only critical point of $x(t)$ occurs at $t = 1$ and $x'(t)$ changes from negative to positive at this time.
- Unsupported positions $x(1)$ and/or $y(1)$ do not earn the third (or fourth) point(s).
- Writing $x(1) = \int_0^1 x'(t) - 2 = -2.603511$ does not earn the third (or fourth) point, because the missing dt makes this statement unclear or false. However, $x(1) = -2 + \int_0^1 x'(t) = -2.603511$ does earn the third point, because it is not ambiguous. Similarly, for $y(1)$.
- For the fourth point the coordinates of the leftmost point do not have to be written as an ordered pair as long as they are labeled as the x- and y-coordinates.
- To earn the last point a response must verify that the particle moves to the right of its initial position (as well as moves to the right for all $t > 1$). Note that there are several ways to demonstrate this.
- Degree mode: y-coordinate $= 5.008$ (or 5.007). (See degree mode statement in part (a).)

Total for part (c) 5 points

Total for question 2 9 points

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Answer QUESTION 2 parts (a) and (b) on this page.

Response for question 2(a)

speed $= \sqrt{(x'(t))^2 + (y'(t))^2}$

$\sqrt{\left((1.2-1)e^{(1.2)^2}\right)^2 + \left(\sin\left((1.2)^{1.25}\right)\right)^2}$

accel: $\langle (t-1)(2te^{t^2}) + (1)(e^{t^2}),\ \cos(t^{1.25}) \cdot 1.25t^{.25} \rangle$

speed $= \boxed{1.271}$

at $t = 1.2$

accel $= \langle 6.247,\ .405 \rangle$

Response for question 2(b)

$\int_0^{1.2} \sqrt{\left((t-1)e^{t^2}\right)^2 + \left(\sin\left(t^{1.25}\right)\right)^2}\, dt$

$= \boxed{1.010}$

Answer QUESTION 2 part (c) on this page.

Response for question 2(c)

$x'(t)$ is negative on $(0,1)$ and is positive on $(1,\infty)$. $x'(1)=0$. Since $x'(t)$ goes from negative to positive at $t=1$ and $x'(1)=0$, $x(t)$ has a rel. minimum at $t=1$. Since $x(t)$ decreases until $t=1$ and always increases afterwards, $x(1)$ is an abs. min /that is the location of furthest to the left.

$$-2+\int_0^1 (t-1)e^{t^2}\,dt = x(1)$$

$$x(1)=-2.604$$

$$5+\int_0^1 (\sin(t^{1.25}))\,dt = y(1)$$

$$y(1)=5.410$$

$$\boxed{(-2.604,\ 5.410)}$$

There is no point farthest to the right because $x'(t)$ increases towards infinity on $t\geq 0$, meaning there is no abs max of $x(t)$ (since $x(t)$ will be increasing towards ∞ on $t>1$). thus This means the particle will continue to the right for $t>1$, creating no furthest right point.

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Answer QUESTION 2 parts (a) and (b) on this page.

Response for question 2(a)

a. $\sqrt{((t-1)e^{t^2})^2 + \sin(t^{1.25})^2}$, $t = 1.2$

$\sqrt{(0.844139)^2 + (0.95084775)^2}$

$= 1.271$

Response for question 2(b)

b. $\int_0^{1.2} \sqrt{((t-1)e^{t^2})^2 + (\sin(t^{1.25}))^2}\, dt$

$= 1.010$

Answer QUESTION 2 part (c) on this page.

Response for question 2(c)

$x'(t)$ = velocity of x

$y'(t)$ = velocity of y

c. farthest left, minimum x position

$(t-1)e^{t^2} = 0$

Critical point: $t=1$ is a minimum because $x''(1) = 2.718 > 0$.

$x(1) = -2 + \int_0^1 x'(t)\,dt$

$= -2.604$

$y(1) = 5 + \int_0^1 y'(t) = 5.405$

Sample 2C

Answer QUESTION 2 parts (a) and (b) on this page.

Response for question 2(a)

$$S = \sqrt{\left((1.2-1)e^{(1.2)^2}\right)^2 + \left(\sin\left(1.2^{1.25}\right)\right)^2}$$

$$\text{Acc.} = \langle 6.2467,\ 0.4051 \rangle$$

Response for question 2(b)

$$\text{TDT} = \int_0^{1.2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= 1.0098$$

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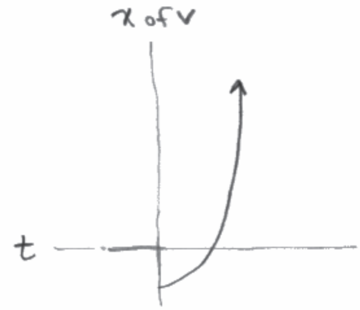
Answer QUESTION 2 part (c) on this page.

Response for question 2(c)

Right − Left → use $x(t)$

$(t-1)e^{t^2}$ @ $t=0$

$x=-2$



x componet of velocity is negative from $t=0$ to $t=1$ therefore since x starts at −2 the furthest left point will be at $t=1$, (−2.6035, 5.4105)

There is no point at which the particle is furthest to the left because the x value continues to increase to infinity. when $t \geq 0$

Question 2

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem a particle moves in the xy-plane with position $(x(t), y(t))$ and velocity $\left\langle (t-1)e^{t^2}, \sin\left(t^{1.25}\right)\right\rangle$; the position is $(-2, 5)$ at time $t = 0$.

In part (a) students were asked to find the speed and the acceleration vector of the particle at time $t = 1.2$. A correct response would show the speed setup, $\sqrt{(x'(1.2))^2 + (y'(1.2))^2}$, and the acceleration setup, $\langle x''(1.2), y''(1.2)\rangle$, and then use a graphing calculator to find both values.

In part (b) students were asked to determine the total distance traveled by the particle over the time interval $0 \le t \le 1.2$. A correct response would present the integral $\int_0^{1.2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$ and determine the numerical value using a graphing calculator.

In part (c) students were asked to find the coordinates of the point (for $t \ge 0$) when the particle is farthest to the left. They were also asked to explain why there is no point at which the particle is farthest to the right. A correct response would determine that the particle changes direction when $x'(t)$ changes sign at time $t = 1$, by setting $x'(t) = 0$. Because $x'(t) < 0$ for $0 < t < 1$ and $x'(t) > 0$ for $t > 1$, the left-most position of the particle would be computed by adding the initial position to the net change, found by integrating the velocity function from $t = 0$ to $t = 1$, for each coordinate position of the particle. Finally, the response should argue that the particle's initial x-coordinate at time $t = 0$ is to the right of the particle's position at time $t = 1$, and from this time on, the particle is moving to the right. Therefore, there is no point at which the particle is farthest to the right.

Sample: 2A
Score: 9

The response earned 9 points: 2 points in part (a), 2 points in part (b), and 5 points in part (c). In part (a) the response earned the first point with the radical expression on the second line. Note that the response continues by simplifying and rounding correctly to the boxed answer of 1.271. While simplifying is not necessary, given that it is presented, it must be correct. The response earned the second point on the last line with the boxed answer of $\langle 6.247, .405\rangle$ and the supporting work found in the vector expression $\left\langle (t-1)\left(2te^{t^2}\right) + (1)e^{t^2}, \cos\left(t^{1.25}\right)\cdot 1.25t^{.25}\right\rangle$ on the lines above.

In part (b) the response earned the first point on the first line with the integrand, noting that it is contained within a definite integral with numeric limits. As the limits on the integral are correct and the boxed answer 1.010 is correct, the response earned the second point. In part (c) the response earned the first and second points by noting that "$x'(t)$ is negative on $(0, 1)$ and is positive on $(1, \infty)$." The second point is reinforced on lines two through six by explaining that due to the fact that $x(t)$ decreases until $t = 1$ and always increases afterward, it follows that $x(1)$ is an absolute minimum. The response earned the third point with the equations $-2 + \int_0^1 (t-1)e^{t^2}\, dt = x(1)$ and $x(1) = -2.604$. The response earned the fourth point with the boxed answer $(-2.604, 5.410)$ and the supporting equation $5 + \int_0^1 \left(\sin\left(t^{1.25}\right)\right) dt = y(1)$ two lines above. The response earned the fifth point in the final paragraph by noting that "$x'(t)$ increases toward infinity on $t > 1$" and correctly concluding that "there is no abs max of $x(t)$ (since $x(t)$ will thus be increasing towards ∞ on $t > 1$)."

Question 2 (continued)

Sample: 2B
Score: 6

The response earned 6 points: 1 point in part (a), 2 points in part (b), and 3 points in part (c). In part (a) the response earned the first point with the first two lines. The third line correctly simplifies the expression in the second line, while the first line supports the work leading to the answer. The response did not earn the second point as there is no answer or work presented for the acceleration vector. In part (b) the response earned the first point in the first line with the integrand, as it is within a definite integral. The response earned the second point with correct limits on the integral and the answer 1.010. In part (c) the response earned the first point with the equation on the third line. The response goes on to make an argument that the x-coordinate is minimized due to the fact that $x''(t) > 0$. This, however, is a local argument. Thus, the response did not earn the second point. The response earned the third point with the equations on the fifth and seventh lines. The response earned the fourth point with the correct value of $x(1)$ and with the equation on the last line. The response did not earn the final point as no answer or argument is given.

Sample: 2C
Score: 3

The response earned 3 points: 1 point in part (a), 2 points in part (b), and no points in part (c). In part (a) the response earned the first point with the radical expression on the first line. While the vector presented on the second line is correct, there is no supporting work; therefore, the response did not earn the second point. In part (b) the response earned the first point on the first line with an appropriate integrand within a definite integral with numeric limits. As the limits of this definite integral are correct, the response earned the second point on the second line with the answer 1.0098. In part (c) the response did not earn the first point. While the response notes in the middle paragraph that the "x componet of velocity is negative from $t = 0$ to $t = 1$," no connection is made between the x-component of "velocity" and $x'(t)$. The response did not earn the second point as the argument given is local and does not support the particle being furthest to the left when $t = 1$. While the coordinates presented at the end of the middle paragraph are correct for the point at $t = 1$, the response earned neither the third nor the fourth points as no supporting work for either of these coordinates is presented. The response did not earn the fifth point because the statement "the x value continues to increase to infinity when $t \geq 0$" contradicts that $x(t)$ is decreasing from $t = 0$ to $t = 1$.

2021

AP CollegeBoard

AP® Calculus BC

Sample Student Responses and Scoring Commentary

Inside:

Free Response Question 3

☑ Scoring Guideline

☑ Student Samples

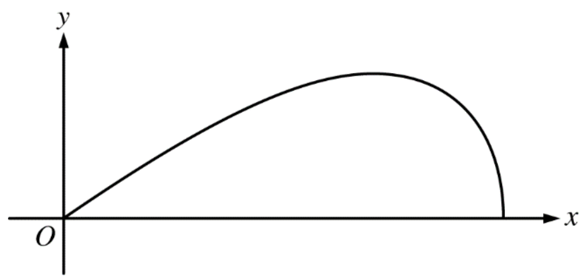
☑ Scoring Commentary

Part B (AB or BC): Graphing calculator not allowed
Question 3 — 9 points

General Scoring Notes

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.



A company designs spinning toys using the family of functions $y = cx\sqrt{4 - x^2}$, where c is a positive constant. The figure above shows the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$, for some c. Each spinning toy is in the shape of the solid generated when such a region is revolved about the x-axis. Both x and y are measured in inches.

	Model Solution	Scoring	
(a)	Find the area of the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$ for $c = 6$.		
	$6x\sqrt{4 - x^2} = 0 \Rightarrow x = 0, x = 2$ Area $= \int_0^2 6x\sqrt{4 - x^2}\, dx$	Integrand	**1 point**
	Let $u = 4 - x^2$. $du = -2x\, dx \Rightarrow -\frac{1}{2}\, du = x\, dx$ $x = 0 \Rightarrow u = 4 - 0^2 = 4$ $x = 2 \Rightarrow u = 4 - 2^2 = 0$ $\int_0^2 6x\sqrt{4 - x^2}\, dx = \int_4^0 6\left(-\frac{1}{2}\right)\sqrt{u}\, du = -3\int_4^0 u^{1/2}\, du = 3\int_0^4 u^{1/2}\, du$ $= 2u^{3/2}\Big\|_{u=0}^{u=4} = 2 \cdot 8 = 16$	Antiderivative	**1 point**
	The area of the region is 16 square inches.	Answer	**1 point**

Scoring notes:

- Units are not required for any points in this question and are not read if presented (correctly or incorrectly) in any part of the response.
- The first point is earned for presenting $cx\sqrt{4 - x^2}$ or $6x\sqrt{4 - x^2}$ as the integrand in a definite integral. Limits of integration (numeric or alphanumeric) must be presented (as part of the definite integral) but do not need to be correct in order to earn the first point.
- If an indefinite integral is presented with an integrand of the correct form, the first point can be earned if the antiderivative (correct or incorrect) is eventually evaluated using the correct limits of integration.
- The second point can be earned without the first point. The second point is earned for the presentation of a correct antiderivative of a function of the form $Ax\sqrt{4 - x^2}$, for any nonzero constant A. If the response has subsequent errors in simplification of the antiderivative or sign errors, the response will earn the second point but will not earn the third point.
- Responses that use u-substitution and have incorrect limits of integration or do not change the limits of integration from x- to u-values are eligible for the second point.
- The response is eligible for the third point only if it has earned the second point.
- The third point is earned only for the answer 16 or equivalent. In the case where a response only presents an indefinite integral, the use of the correct limits of integration to evaluate the antiderivative must be shown to earn the third point.
- The response cannot correct -16 to $+16$ in order to earn the third point; there is no possible reversal here.

Total for part (a) 3 points

(b) It is known that, for $y = cx\sqrt{4 - x^2}$, $\dfrac{dy}{dx} = \dfrac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of c for this spinning toy?

The cross-sectional circular slice with the largest radius occurs where $cx\sqrt{4 - x^2}$ has its maximum on the interval $0 < x < 2$. $\dfrac{dy}{dx} = \dfrac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}} = 0 \Rightarrow x = \sqrt{2}$	Sets $\dfrac{dy}{dx} = 0$	**1 point**
$x = \sqrt{2} \Rightarrow y = c\sqrt{2}\sqrt{4 - \left(\sqrt{2}\right)^2} = 2c$ $2c = 1.2 \Rightarrow c = 0.6$	Answer	**1 point**

Scoring notes:

- The first point is earned for setting $\dfrac{dy}{dx} = 0$, $\dfrac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}} = 0$, or $c\left(4 - 2x^2\right) = 0$.
- An unsupported $x = \sqrt{2}$ does not earn the first point.
- The second point can be earned without the first point but is earned only for the answer $c = 0.6$ with supporting work.

Total for part (b) 2 points

(c) For another spinning toy, the volume is 2π cubic inches. What is the value of c for this spinning toy?

$\text{Volume} = \int_0^2 \pi\left(cx\sqrt{4-x^2}\right)^2\,dx = \pi c^2 \int_0^2 x^2\left(4-x^2\right)\,dx$	Form of the integrand	**1 point**
	Limits and constant	**1 point**
$= \pi c^2 \int_0^2 \left(4x^2 - x^4\right)\,dx = \pi c^2 \left(\frac{4}{3}x^3 - \frac{1}{5}x^5 \Big\|_0^2\right)$	Antiderivative	**1 point**
$= \pi c^2 \left(\frac{32}{3} - \frac{32}{5}\right) = \frac{64\pi c^2}{15}$ $\frac{64\pi c^2}{15} = 2\pi \Rightarrow c^2 = \frac{15}{32} \Rightarrow c = \sqrt{\frac{15}{32}}$	Answer	**1 point**

Scoring notes:

- The first point is earned for presenting an integrand of the form $A\left(x\sqrt{4-x^2}\right)^2$ in a definite integral with any limits of integration (numeric or alphanumeric) and any nonzero constant A. Mishandling the c will result in the response being ineligible for the fourth point.
- The second point can be earned without the first point. The second point is earned for the limits of integration, $x = 0$ and $x = 2$, and the constant π (but not for 2π) as part of an integral with a correct or incorrect integrand.
- If an indefinite integral is presented with the correct constant π, the second point can be earned if the antiderivative (correct or incorrect) is evaluated using the correct limits of integration.
- A response that presents $2 = \int_0^2 \left(cx\sqrt{4-x^2}\right)^2\,dx$ earns the first and second points.
- The third point is earned for presenting a correct antiderivative of the presented integrand of the form $A\left(x\sqrt{4-x^2}\right)^2$ for any nonzero A. If there are subsequent errors in simplification of the antiderivative, linkage errors, or sign errors, the response will not earn the fourth point.
- The fourth point cannot be earned without the third point. The fourth point is earned only for the correct answer. The expression does not need to be simplified to earn the fourth point.

Total for part (c) 4 points

Total for question 3 9 points

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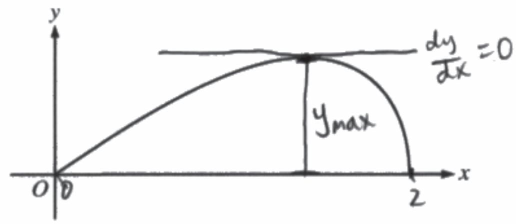
Answer QUESTION 3 part (a) on this page.



Response for question 3(a)

$y = 6x\sqrt{4-x^2} = 0$

$x = 0 \ , \ x = 2$

$A = \int_0^2 6x\sqrt{4-x^2}\,dx$ $\qquad u = 4-x^2$

$du = -2x\,dx$

$A = \int_4^0 -3\sqrt{u}\,du = 3\int_0^4 u^{\frac{1}{2}}\,du = 3\,u^{3/2}\cdot\frac{2}{3}\Big|_0^4$

$A = 2(4^{3/2} - 0^{3/2}) = 2(2^{3} - 0) = 2(8) = 16$

$A = 16$

Answer QUESTION 3 parts (b) and (c) on this page.

Response for question 3(b)

Largest cross section where y is greatest (maximum of y on graph).

Find max:

$\frac{dy}{dx} = \frac{c(4-2x^2)}{\sqrt{4-x^2}} = 0 \rightarrow c(4-2x^2)=0$

$4 = 2x^2$

$x = \sqrt{2}$

At $x=\sqrt{2}$, $y=1.2$ (largest radius of cross-section equals 1.2, which is max y value)

$y = cx\sqrt{4-x^2}$

$1.2 = c\sqrt{2}(\sqrt{4-(\sqrt{2})^2}) = c\sqrt{2}(\sqrt{4-2}) = c\sqrt{2}(\sqrt{2}) = 2c$

$c = \frac{1.2}{2} = 0.6 \rightarrow \boxed{c = 0.6}$

Response for question 3(c)

$V = \pi\int_0^2 y^2\,dx = \pi\int_0^2 (cx\sqrt{4-x^2})^2\,dx = \pi c^2\int_0^2 x^2(4-x^2)\,dx = \pi c^2\int_0^2 (4x^2-x^4)\,dx$

$V = \pi c^2\left[\frac{4x^3}{3} - \frac{x^5}{5}\Big|_0^2\right] = \pi c^2\left[\left(\frac{4(8)}{3} - \frac{(32)}{5}\right) - (0-0)\right]$

$V = \pi c^2\left(\frac{32(5)}{3(5)} - \frac{32(3)}{5(3)}\right) = \pi c^2\left(\frac{2(32)}{15}\right) = \pi c^2\left(\frac{64}{15}\right)$

$2\pi = \pi c^2\left(\frac{64}{15}\right)$

$c^2 = \frac{30}{64} \rightarrow c = \sqrt{\frac{30}{64}} = \boxed{\frac{\sqrt{30}}{8}}$

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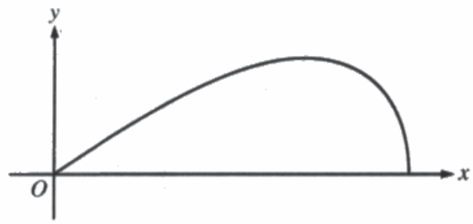
Answer QUESTION 3 part (a) on this page.



Response for question 3(a)

$y = 6x\sqrt{4-x^2}$

$6x\sqrt{4-x^2} = 0$

$x = 0$

$x = 2$

$A = \int_0^2 6x\sqrt{4-x^2}\,dx$

$u = 4-x^2$

$du = -2x\,dx$

$-3du = 6x\,dx$

$u = 4-2^2$, $u = 0$

$u = 4-0^2$, $u = 4$

$A = \int_4^0 -3\sqrt{u}\,du$

$A = -3\int_4^0 \sqrt{u}\,du$

$\frac{1}{2} + \frac{2}{2} = \frac{3}{2}$

$-3\left[\frac{2(u)^{3/2}}{3}\right]\Big|_4^0$

$-3\left[\frac{2(4-x^2)^{3/2}}{3}\right]\Big|_0^2$

$= -3\left[\frac{2(4-2^2)^{3/2}}{3} - \frac{2(4-0^2)^{3/2}}{3}\right]$

$= -3\left(0 - \frac{2(4)^{3/2}}{3}\right)$

$= \frac{6(4)^{3/2}}{3} = 2(4)^{3/2}$

$= 16$

3 3 3 3 3 NO CALCULATOR ALLOWED 3 3 3 3 3

Answer QUESTION 3 parts (b) and (c) on this page.

Response for question 3(b)

$\frac{dy}{dx} = 0$

$\frac{c(4-2x^2)}{\sqrt{4-x^2}} = 0$

$c(4-2x^2) = 0$

$4-2x^2=0$

$4 = 2x^2$

$2 = x^2$

$x = \sqrt{2}$

$1.2 = c\sqrt{2} \times \sqrt{4-(\sqrt{2})^2}$

$= c\sqrt{2} \times \sqrt{4-2}$

$= c\sqrt{2} \times \sqrt{2}$

$1.2 = 2c$

$c = \frac{1.2}{2}$

$c = 0.6$

Response for question 3(c)

$V = \pi \int r^2 dx$

$V = \pi \int_0^2 cx\sqrt{4-x^2}\, dx$

$c\pi \int_0^2 x\sqrt{4-x^2}\, dx = 2\pi$

$u = 4-x^2$

$\frac{du}{-2} = \frac{-2x}{-2} dx$

$du = x\, dx$

$u = 4-2^2 = 0$

$4-0^2 = 4$

$c\pi \int_4^0 \sqrt{u}\, du = 2\pi$

$c\pi \left[\frac{2u^{3/2}}{3} \Big|_4^0 \right] = 2\pi$

$\frac{2(4-x^2)^{3/2}}{3} \Big|_0^2 = \frac{2}{c}$

$0 - \frac{2(4)^{3/2}}{3} = \frac{2}{c}$

$\frac{-16}{3} = \frac{2}{c}$

$c = \frac{2 \times 3}{-16}$

$c = \frac{-6}{16}$

$c = \frac{-3}{8}$

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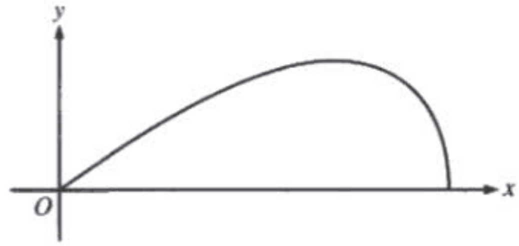
Answer QUESTION 3 part (a) on this page.



Response for question 3(a)

$y = 6x\sqrt{4-x^2}$ $\quad y = 0 = 6x\sqrt{4-x^2}$

$x = 0, 2$

$A = \int_0^2 6x(4-x^2)^{1/2}\,dx$

$A = \frac{1}{2}6\int_0^2 (2x)(4-x^2)^{1/3}\,dx$

$A = 3\left[\frac{2}{3}(4-x^2)^{3/2}\right)\Big|_0^2$

$A = 3\left(\frac{2}{3}(0) - \frac{2}{3}(4)^{3/2}\right)$

$3\left(0 - \frac{16}{3}\right)$

$|-16| = \boxed{16 \text{ inches}^2}$

Answer QUESTION 3 parts (b) and (c) on this page.

Response for question 3(b)

$y = Cx\sqrt{4-x^2}$ $\qquad \frac{dy}{dx} = \frac{C(4-2x^2)}{\sqrt{4-x^2}} = 0$

$4 = 2x^2$

$2 = x^2$

$x = \sqrt{2}$ CN

$\frac{dy}{dx}$ +++++++ | - - - - ($\sqrt{2}$)

$y(\sqrt{2}) = 1.2 = C\sqrt{2}\sqrt{4-2}$

$1.2 = C\sqrt{2}\sqrt{2}$

$1.2 = 2C$

$C = .6$

Response for question 3(c)

$y = Cx\sqrt{4-x^2} = 0$

$x = 0, 2$

$V = 2\pi \int_0^2 x\left(Cx\sqrt{4-x^2}\right) dx$

$V = 2\pi = 2\pi \int_0^2 x\left(Cx\sqrt{4-x^2}\right) dx$

$\int_0^2 x\left(Cx\sqrt{4-x^2}\right) dx = ?$

?

Question 3

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem a company designs spinning toys using various functions of the form $y = cx\sqrt{4 - x^2}$, where c is a positive constant. A graph of the region in the first quadrant bounded by the x-axis and this function for some c is given and students were told that the spinning toys are in the shape of the solid generated when this region is revolved around the x-axis. Both x and y are measured in inches.

In part (a) students were asked to find the area of the region in the first quadrant bounded by the x-axis and the region $y = cx\sqrt{4 - x^2}$ for $c = 6.$ A correct response will set up the definite integral $\int_0^2 6x\sqrt{4 - x^2}\, dx$ and use the method of substitution to evaluate the integral to obtain an area of 16.

In part (b) students were told that for $y = cx\sqrt{4 - x^2}$, $\frac{dy}{dx} = \frac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}}.$ They were also told that for a particular spinning toy the radius of the largest cross-sectional circular slice is 1.2 inches and were asked to find the value of c for this particular spinning toy. A correct response will solve $\frac{dy}{dx} = 0$ to find that the largest radius occurs when $x = \sqrt{2}.$ Then using this value of x in the equation $y = cx\sqrt{4 - x^2} = 1.2,$ the value of c is found to be $0.6.$

In part (c) students were told that for another spinning toy, the volume is 2π cubic inches. They were asked to find the value of c for this spinning toy. A correct response would set up the volume of the toy as the integral $\int_0^2 \pi\left(cx\sqrt{4 - x^2}\right)^2 dx,$ evaluate this integral, and set the value equal to $2\pi.$ Solving the resulting equation for c results in $c = \sqrt{\frac{15}{32}}.$

Sample: 3A
Score: 9

The response earned 9 points: 3 points in part (a), 2 points in part (b), and 4 points in part (c). In part (a) the response presents the correct integrand in a definite integral and earned the first point. The antiderivative of $3u^{3/2} \cdot \frac{2}{3}$ with the definition $u = 4 - x^2$ is correct and earned the second point. The response has the correct answer and earned the third point. In part (b) the response earned the first point for stating $\frac{dy}{dx} = \frac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}} = 0.$ The answer is correct, and the response earned the second point. In part (c) the response presents y^2 as the integrand of a definite integral and earned the first point. Note that because $y = cx\sqrt{4 - x^2}$ is given in the statement of the problem, a response can reference the function by using y for the first point. The limits and constant are correct and earned the second point. The antiderivative is correct and earned the third point. The response is eligible for the fourth point. The answer is correct and earned the fourth point. Note that $\frac{\sqrt{30}}{8} = \sqrt{\frac{15}{32}}.$

Question 3 (continued)

Sample: 3B
Score: 6

The response earned 6 points: 3 points in part (a), 2 points in part (b), and 1 point in part (c). In part (a) the response presents the correct integrand in a definite integral and earned the first point. The antiderivative $-3\left[\frac{2(u)^{3/2}}{3}\right]$ with the definition $u = 4 - x^2$ is correct and earned the second point. Note that the sign of the antiderivative is consistent with the limits of integration. The response is eligible for the third point. The answer is correct and earned the third point. Note that the substitution of $u = 4 - x^2$ after finding the antiderivative and using the limits of $x = 0$ and $x = 2$ is not necessary to evaluate the antiderivative. In part (b) the response earned the first point by stating $\frac{dy}{dx} = 0.$ The answer is correct and earned the second point. In part (c) the integrand is not of the correct form and the response did not earn the first point. The limits and constant are correct and earned the second point. Because the integrand is not of the correct form, the response is not eligible for and did not earn the third point. Without earning the third point, the response is not eligible for and did not earn the fourth point.

Sample: 3C
Score: 3

The response earned 3 points: 1 point in part (a), 2 points in part (b), and no points in part (c). In part (a) the response presents the correct integrand in a definite integral and earned the first point. The antiderivative presented is incorrect because the sign of the antiderivative is incorrect, and the response did not earn the second point. The response is not eligible for and did not earn the third point. In part (b) the response earned the first point by stating $\frac{dy}{dx} = \frac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}} = 0.$ The answer is correct, and the response earned the second point. In part (c) the integrand presented is not of the correct form and the response did not earn the first point. The constant 2π is incorrect and the response did not earn the second point. Without an integrand of the correct form, the response is not eligible for the third point and is not eligible for the fourth point. The response did not earn the third point and did not earn the fourth point.

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AP® Calculus BC

Sample Student Responses and Scoring Commentary

Inside:

Free Response Question 4

☑ Scoring Guideline

☑ Student Samples

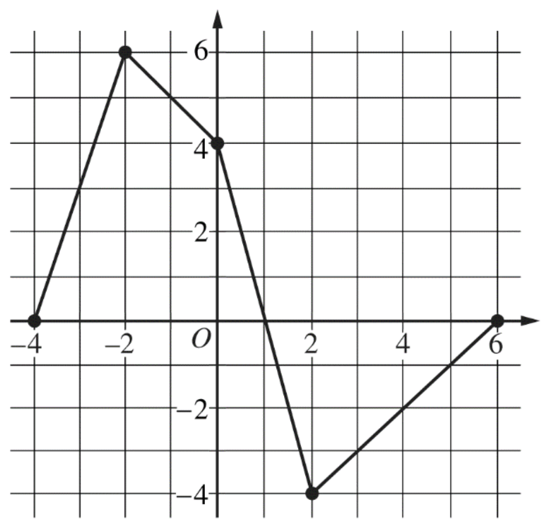
☑ Scoring Commentary

Part B (AB or BC): Graphing calculator not allowed
Question 4 — 9 points

General Scoring Notes

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.



Graph of f

Let f be a continuous function defined on the closed interval $-4 \le x \le 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t)\,dt$.

Model Solution	Scoring	
$G'(x) = f(x)$ in any part of the response.	$G'(x) = f(x)$	**1 point**

Scoring notes:

- This "global point" can be earned in any one part. Expressions that show this connection and therefore earn this point include: $G' = f$, $G'(x) = f(x)$, $G''(x) = f'(x)$ in part (a), $G'(3) = f(3)$ in part (b), or $G'(2) = f(2)$ in part (c).

Total **1 point**

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

$G'(x) = f(x)$ The graph of G is concave up for $-4 < x < -2$ and $2 < x < 6$, because $G' = f$ is increasing on these intervals.	Answer with reason	**1 point**

Scoring notes:

- Intervals may also include one or both endpoints.

Total for part (a) **1 point**

(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

$P'(x) = G(x) \cdot f'(x) + f(x) \cdot G'(x)$ $P'(3) = G(3) \cdot f'(3) + f(3) \cdot G'(3)$	Product rule	**1 point**
Substituting $G(3) = \int_0^3 f(t)\, dt = -3.5$ and $G'(3) = f(3) = -3$ into the above expression for $P'(3)$ gives the following:	$G(3)$ or $G'(3)$	**1 point**
$P'(3) = -3.5 \cdot 1 + (-3) \cdot (-3) = 5.5$	Answer	**1 point**

Scoring notes:

- The first point is earned for the correct application of the product rule in terms of x or in the evaluation of $P'(3)$. Once earned, this point cannot be lost.
- The second point is earned by correctly evaluating $G(3) = -3.5$, $G'(3) = -3$, or $f(3) = -3$.
- To be eligible to earn the third point a response must have earned the first two points.
- Simplification of the numerical value is not required to earn the third point.

Total for part (b) **3 points**

(c) Find $\lim_{x \to 2} \frac{G(x)}{x^2 - 2x}$.

$\lim_{x \to 2} \left(x^2 - 2x\right) = 0$ Because G is continuous for $-4 \le x \le 6$, $\lim_{x \to 2} G(x) = \int_0^2 f(t)\,dt = 0$. Therefore, the limit $\lim_{x \to 2} \frac{G(x)}{x^2 - 2x}$ is an indeterminate form of type $\frac{0}{0}$.	Uses L'Hospital's Rule	**1 point**
Using L'Hospital's Rule, $\lim_{x \to 2} \frac{G(x)}{x^2 - 2x} = \lim_{x \to 2} \frac{G'(x)}{2x - 2}$ $= \lim_{x \to 2} \frac{f(x)}{2x - 2} = \frac{f(2)}{2} = \frac{-4}{2} = -2$	Answer with justification	**1 point**

Scoring notes:

- To earn the first point the response must show $\lim_{x \to 2} \left(x^2 - 2x\right) = 0$ and $\lim_{x \to 2} G(x) = 0$ and must show a ratio of the two derivatives, $G'(x)$ and $2x - 2$. The ratio may be shown as evaluations of the derivatives at $x = 2$, such as $\frac{G'(2)}{2}$.
- To earn the second point the response must evaluate correctly with appropriate limit notation. In particular the response must include $\lim_{x \to 2} \frac{G'(x)}{2x - 2}$ or $\lim_{x \to 2} \frac{f(x)}{2x - 2}$.
- With any linkage errors (such as $\frac{G'(x)}{2x - 2} = \frac{f(2)}{2}$), the response does not earn the second point.

Total for part (c) 2 points

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

$G(2) = \int_0^2 f(t)\,dt = 0$ and $G(-4) = \int_0^{-4} f(t)\,dt = -16$ Average rate of change $= \dfrac{G(2) - G(-4)}{2 - (-4)} = \dfrac{0 - (-16)}{6} = \dfrac{8}{3}$	Average rate of change	**1 point**
Yes, $G'(x) = f(x)$ so G is differentiable on $(-4, 2)$ and continuous on $[-4, 2]$. Therefore, the Mean Value Theorem applies and guarantees a value c, $-4 < c < 2$, such that $G'(c) = \frac{8}{3}$.	Answer with justification	**1 point**

Scoring notes:

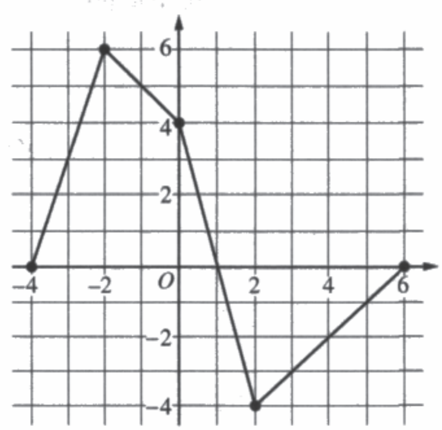
- To earn the first point a response must present at least a difference and a quotient and a correct evaluation. For example, $\frac{0+16}{6}$ or $\frac{G(2) - G(-4)}{6} = \frac{16}{6}$.
- Simplification of the numerical value is not required to earn the first point, but any simplification must be correct.
- The second point can be earned without the first point if the response has the correct setup but an incorrect or no evaluation of the average rate of change. The Mean Value Theorem need not be explicitly stated provided the conditions and conclusion are stated.

Total for part (d) 2 points

Total for question 4 9 points

4 4 4 4 4 NO CALCULATOR ALLOWED 4 4 4 4 4

Answer QUESTION 4 parts (a) and (b) on this page.



Graph of f

Response for question 4(a)

$G'(x) = f(x)$

$G''(x) = f'(x)$

On $(-4,-2)$ and $(2,6)$, $G(x)$ is concave up because $f(x)$ (which is equal to $G'(x)$) has a positive slope / is increasing.

Response for question 4(b)

$P'(x) = G'(x)\,f(x) + f'(x)\,G(x)$

$P'(3) = G'(3)\,f(3) + f'(3)\,G(3)$

$G'(x) = f(x)$

$G'(3) = f(3) = -3$

$G(3) = \int_0^3 f(t)\,dt = -\frac{7}{2}$

$P'(3) = (-3)(-3) + (1)\left(-\frac{7}{2}\right)$

4 4 4 4 4 NO CALCULATOR ALLOWED 4 4 4 4 4

Answer QUESTION 4 parts (c) and (d) on this page.

Response for question 4(c)

$\lim_{x\to 2} G(x) = \lim_{x\to 2} (x^2-2x) = 0$ Must use l'hôpital's rule

$\llcorner \int_0^2 f(t)\,dt = 0$

$\lim_{x\to 2} \frac{G'(x)}{2x-2} = \frac{f(2)}{4-2} = \frac{-4}{2}$

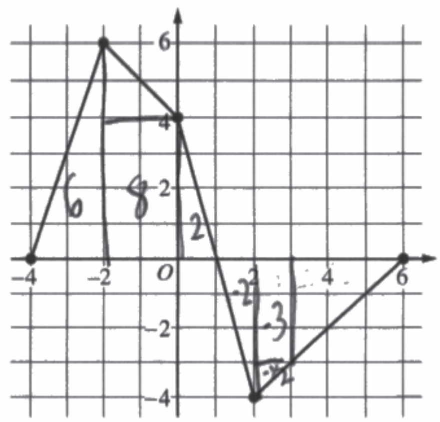
Response for question 4(d)

$\int_0^2 f(t)\,dt$ $\int_0^{-4} f(t)\,dt = -(3+9+3+1) = -16$

$\text{AROC of } G = \frac{G(2)-G(-4)}{2-(-4)} = \frac{0-(-16)}{2+4} = \frac{16}{6} = \frac{8}{3}$

The meanvalue theorem does guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to average rate of change. This is because ~~$G'(x) = f(t)$~~ and ~~x=t~~ exists for all values, ~~$-4<x<2$~~, meaning that $G(x)$ is continuous on the closed interval and differentiable on the open interval.

Answer QUESTION 4 parts (a) and (b) on this page.



Graph of f

Response for question 4(a)

The graph of G is concave up on the intervals $(-4, -2) \cup (2, 6)$ because G is concave up when G' is increasing and $G'(x) = f(x)$.

Response for question 4(b)

$(4)(1)(\frac{1}{2}) = \cancel{2}\ \cancel{-2} - 3 - \frac{1}{2} = -3 - \frac{1}{2} = -\frac{6}{2} - \frac{1}{2} = -\frac{5}{2}$

$P(x) = G(x) \cdot f(x)$

$P'(x) = G'(x) \cdot f(x) + f'(x) \cdot G(x)$

$P'(3) = G'(3) \cdot f(3) + f'(3) \cdot G(3)$

$= (-3) \cdot (-3) + (-4) \cdot (-\frac{5}{2})$

$= 9 + 10 = 19$

$-\frac{5}{\cancel{2}_1} \cdot -\frac{\cancel{4}^2}{1} = 10$

$P'(3) \approx 19$

Answer QUESTION 4 parts (c) and (d) on this page.

Response for question 4(c)

$\lim_{x \to 2} \frac{G(x) \to 0}{x^2 - 2x \to 0}$ L'hopital rule

$\lim_{x \to 2} \frac{G'(x)}{2x-2} = \frac{G'(2)}{2(2)-2} = \frac{-4}{2} = -2$

Response for question 4(d)

$\frac{1}{6}\int_{-4}^{2} -4 = -\frac{4}{6} = -\frac{2}{3}$

$\frac{G(2) - G(-4)}{-4-2} = \frac{0-(-(6+8+2))}{-6} = \frac{16}{-6} = -\frac{8}{3}$

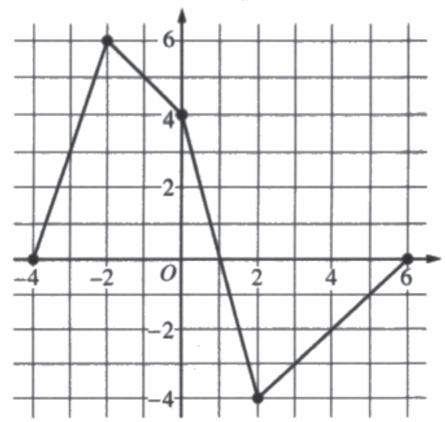
The MVT doesn't guarantee the value of c, $-4 < c < 2$ for which $G'(c)$ is equal to the average rate of change of G because the values of the average rate of change are different since G' is the derivative G,

$\frac{G'(2) - G'(-4)}{-4-2} = \frac{-4-0}{-6} = \frac{-4}{-6} = \frac{2}{3}$

$\frac{1}{2-4}\int_{-4}^{2} G(x)dx = \frac{1}{6} \cdot 16 = \frac{16}{6} = \frac{8}{3}$

4 4 4 4 4 NO CALCULATOR ALLOWED 4 4 4 4 4

Answer QUESTION 4 parts (a) and (b) on this page.



Graph of f

Response for question 4(a)

$G(x) = \int_0^x f(t)\,dt$

$G(x) = \int f(x)\,dx$

$f(x) = G'(x)$

Because f(x) is the antiderivative of G(x) that means that f(x) = G'(x). Therefore when the graph f(x) = increasing, then that means G(x) = concave up. G is concave up on the intervals (−4, −2) ∪ (2, 6) b/c f is increasing.

Response for question 4(b)

$p(x) = G(x) \cdot f(x)$

$p'(x) = G'(x) \cdot f(x) + f'(x) \cdot G(x)$

$p'(3) = G'(2) \cdot f(2) + f'(3) \cdot G(2)$

$p'(3) = -3 \cdot -3 + -3 \cdot -3$

$p'(3) = 9 + 9$

$p'(3) = 18$

4 4 4 4 4 NO CALCULATOR ALLOWED 4 4 4 4 4

Answer QUESTION 4 parts (c) and (d) on this page.

Response for question 4(c)

$\lim_{x\to 2} \frac{G(x)}{x^2-2x} \overset{\text{L'HOP}}{\to} \frac{G'(x)}{2x-2} = \frac{f(x)}{2x-2}$

$G'(x) = f(x)$

$= \frac{f(2)}{2(2)-2}$

$= \frac{-4}{2}$

$= -2$

Response for question 4(d)

$G(x) = \int_{-4}^{2} f(x)\,dx$

$G'(x) = f(x)\Big|_{-4}^{2}$

$= f(2) - f(-4)$

$= -4 - 0$

$= \boxed{-4}$

$-4 < c < 2 \quad G'(c) = -4?$

NO b/c c cannot equal -4, it can only be greater than it. Therefore the Mean Value Theorem does not guarantee a value.

Question 4

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem the graph of a piecewise linear continuous function f for $-4 \le x \le 6$ is provided. It is also given that $G(x) = \int_0^x f(t)\,dt$.

In part (a) students were asked to provide the open intervals on which the graph of G is concave up. A correct response would use the Fundamental Theorem of Calculus to note that $G' = f$, and then report the two intervals where $G' = f$ is increasing.

In part (b) the function $P(x) = G(x) \cdot f(x)$ is defined and students were asked to find $P'(3)$. A correct response would use the product rule to find an expression for $P'(x)$, then use the graph of f to find numerical values of $f(3)$ and $f'(3)$, and use the Fundamental Theorem of Calculus to find $G(3)$ and $G'(3)$. The response would substitute these values into the expression for $P'(x)$ to provide the value of $P'(3)$.

In part (c) students were asked to find $\lim_{x \to 2} \frac{G(x)}{x^2 - 2x}$. A correct response would use L'Hospital's Rule to find the limit after verifying that the limits of both the numerator and denominator are zero.

In part (d) students were asked to find the average rate of change of G on the interval $[-4, 2]$ and whether the Mean Value Theorem guarantees a value c, $-4 < c < 2$, with $G'(c)$ equal to this average rate of change. A correct response would determine the average rate of change as a difference quotient, $\frac{G(2) - G(-4)}{2 - (-4)}$, with values $G(2) = 0$ and $G(-4) = -16$ found as areas under the graph of f. The response should then conclude that the Mean Value Theorem does guarantee such a value of c because $G' = f$ is differentiable and, therefore, continuous on the given interval.

Sample: 4A
Score: 9

The response earned 9 points: 1 global point, 1 point in part (a), 3 points in part (b), 2 points in part (c), and 2 points in part (d). The global point was earned in the first line of part (a) with the statement $G'(x) = f(x)$. In part (a) the response earned the point with the presentation of the correct intervals and the reason "$f(x)$ which is equal to $G'(x)$ has a positive slope/is increasing." In part (b) the response earned the first point with the correct product rule presentation in the first line. The second point was earned for the correct values for both $G'(3)$ and $G(3)$ although only one correct value was necessary for this point. The third point was earned with the expression $(-3)(-3)+(1)\left(-\frac{7}{2}\right)$. Simplification of this expression is not necessary. In part (c) the response earned the first point with the extended equation of limits in the first line and the ratio of derivatives in the second line. The second point was earned with the correct ratio of derivatives accompanied by limit notation and the correct (unsimplified) answer. In part (d) the response earned the first point for a valid attempt to calculate the average rate of change of G and a correct result. The second point was earned with the answer, "The mean value theorem does guarantee a value c," and the statement that $G(x)$ is both differentiable and continuous.

Question 4 (continued)

Sample: 4B
Score: 6

The response earned 6 points: 1 global point, 1 point in part (a), 2 points in part (b), 2 points in part (c), and no points in part (d). The global point was earned in part (a) with the statement $G'(x) = f(x)$. In part (a) the response earned the point with the presentation of the correct intervals and the reason that G' is increasing. In part (b) the response earned the first point with the correct product rule in the second line. The second point was earned with the value of $G'(3)$ as -3 in the fourth line. Note that the incorrect value of $G(3)$ did not affect this point because only one correct value of $G(3)$ or $G'(3)$ is required. The third point was not earned due to the incorrect final answer. In part (c) the first point was earned with the arrows pointing from the numerator and denominator to the value 0 and by the ratio of derivatives. The second point was earned with the correct ratio of derivatives accompanied by limit notation and the correct answer. In part (d) the first point was not earned because the average rate of change presented is not correct (denominator should be $2 - (-4)$). Because this is not a valid average rate of change form, the response is not eligible for the second point.

Sample: 4C
Score: 4

The response earned 4 points: 1 global point, 1 point in part (a), 2 points in part (b), no points in part (c), and no points in part (d). The global point was earned in the third line of part (a) with the statement $f(x) = G'(x)$. In part (a) the response earned the point with the presentation of the correct intervals and the reason "when $f(x) =$ increasing, then $G(x) =$ concave up." In part (b) the response earned the first point with the correct product rule in the second line. The second point was earned for the correct value for $G(3)$. The value for $G'(3)$ is incorrect, but only one correct value is necessary for this point. The third point was not earned because the final answer is incorrect. In part (c) the response did not earn the first point because there is no evidence of $\lim_{x \to 2} G(x) = 0$ or $\lim_{x \to 2} \left(x^2 - 2x\right) = 0$ given. The second point was not earned because the ratio of derivatives does not have limit notation. In part (d) the response did not earn the first point because there is not an attempt to calculate the average rate of change of G. The response is not eligible for the second point.

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Sample Student Responses and Scoring Commentary

Inside:

Free Response Question 5

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Part B (BC): Graphing calculator not allowed
Question 5 — 9 points

General Scoring Notes
Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.

Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$. It can be shown that $f''(1) = 4$.

Model Solution	Scoring	
(a) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(2)$.		
$f'(1) = \left.\frac{dy}{dx}\right\rvert_{(x,\,y)=(1,\,4)} = 4 \cdot (1 \ln 1) = 0$ The second-degree Taylor polynomial for f about $x = 1$ is $f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 = 4 + 0(x-1) + \frac{4}{2}(x-1)^2$.	Polynomial	**1 point**
$f(2) \approx 4 + 2(2-1)^2 = 6$	Approximation	**1 point**

Scoring notes:

- The first point is earned for $4 + \frac{4 \cdot \ln 1}{1!}(x-1)^1 + \frac{4}{2!}(x-1)^2$ or any correctly simplified equivalent expression. A term involving $(x-1)$ is not necessary. The polynomial must be written about (centered at) $x = 1$.
- If the first point is earned, the second point is earned for just "6" with no additional supporting work.
- If the polynomial is never explicitly written, the first point is not earned. In this case, to earn the second point supporting work of at least "$4 + 2(1)$" is required.

Total for part (a) 2 points

(b) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(2)$. Show the work that leads to your answer.

$f(1.5) \approx f(1) + 0.5 \cdot \left.\dfrac{dy}{dx}\right\|_{(x,\,y)=(1,\,4)} = 4 + 0.5 \cdot 0 = 4$ $f(2) \approx f(1.5) + 0.5 \cdot \left.\dfrac{dy}{dx}\right\|_{(x,\,y)=(1.5,\,4)}$	Euler's method with two steps	**1 point**
$\approx 4 + 0.5 \cdot 4 \cdot (1.5 \ln 1.5) = 4 + 3 \ln 1.5$	Answer	**1 point**

Scoring notes:

- The first point is earned for two steps (of size 0.5) of Euler's method, with at most one error. If there is any error, the second point is not earned.
- To earn the first point a response must contain two Euler steps, $\Delta x = 0.5$, use of the correct expression for $\dfrac{dy}{dx}$, and use of the initial condition $f(1) = 4$.
    - The two Euler steps may be explicit expressions or may be presented in a table. Here is a minimal example of a (correctly labeled) table.

x	y	$\Delta y = \dfrac{dy}{dx} \cdot \Delta x$ or $\Delta y = \dfrac{dy}{dx} \cdot (0.5)$
1	4	0
1.5	4	$3 \ln 1.5$
2	$4 + 3 \ln 1.5$	

    - Note: In the presence of the correct answer, such a table does not need to be labeled in order to earn both points. In the presence of an incorrect answer, the table must be correctly labeled for the response to earn the first point.
- A single error in computing the approximation of $f(1.5)$ is not considered a second error if that incorrect value is imported correctly into an approximation of $f(2)$.
- Both points are earned for "$4 + 0.5 \cdot 0 + 0.5 \cdot 4 \cdot (1.5 \ln 1.5)$" or "$4 + 0.5 \cdot 4 \cdot (1.5 \ln 1.5)$".
- Both points are earned for presenting the ordered pair $(2,\ 4 + 3 \ln 1.5)$ with sufficient supporting work.

Total for part (b) **2 points**

(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$.

$\frac{1}{y}\,dy = x \ln x\,dx$	Separation of variables	**1 point**
Using integration by parts, $u = \ln x \quad du = \frac{1}{x}\,dx$ $dv = x\,dx \quad v = \frac{x^2}{2}$ $\int x \ln x\,dx = \frac{x^2}{2} \cdot \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x}\,dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$	Antiderivative for $x \ln x$	**1 point**
$\ln\lvert y\rvert = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$	Antiderivative for $\frac{1}{y}$	**1 point**
$\ln 4 = 0 - \frac{1}{4} + C \Rightarrow C = \ln 4 + \frac{1}{4}$	Constant of integration and uses initial condition	**1 point**
$y = e^{\left(\frac{x^2 \ln x}{2} - \frac{x^2}{4} + \ln 4 + \frac{1}{4}\right)}$ Note: This solution is valid for $x > 0$.	Solves for y	**1 point**

Scoring notes:

- A response with no separation of variables earns 0 out of 5 points. If an error in separation results in one side being correct ($\frac{1}{y}\,dy$ or $x \ln x\,dx$), the response is only eligible to earn the corresponding antiderivative point.
- The third point (antiderivative of $\frac{1}{y}$) can be earned for either $\ln y$ or $\ln\lvert y\rvert$.
- A response with no constant of integration can earn at most 3 out of 5 points.
- A response is eligible for the fourth point if it has earned the first point and at least 1 of the 2 antiderivative points.
- A response earns the fourth point by correctly including the constant of integration in an equation and then replacing x with 1 and y with 4.
- A response is eligible for the fifth point only if it has earned the first 4 points.

Total for part (c) **5 points**

Total for question 5 **9 points**

Answer QUESTION 5 parts (a) and (b) on this page.

Response for question 5(a)

$\frac{dy}{dx} = 4 \cdot \ln 1$

$P_2(x) = 4 + \ln 1 \cdot (x-1) + \frac{4(x-1)^2}{2!}$ $\quad P_2(x) = 4 + \frac{4(x-1)^2}{2!}$

$P_2(2) = 4 + \frac{4(2-1)^2}{2!} = 4 + 2 = \boxed{6}$

Response for question 5(b)

$\frac{dy}{dx} = 4 \cdot 1.5 \ln 1.5$

$6 \ln 1.5$

(x, y)	Δx	$\frac{dy}{dx}$	Δy
$(1, 4)$	$.5$	0	0
$(1.5, 4)$	$.5$	$6 \ln 1.5$	$3 \ln 1.5$
$(2, \ )$			

$\boxed{f(2) = 4 + 3 \ln 1.5}$

Answer QUESTION 5 part (c) on this page.

Response for question 5(c)

$$\frac{dy}{dx} = y \cdot x\ln x$$

$$\frac{1}{y}dy = x\ln x\,dx$$

$$\int \frac{1}{y}dy = \int x\ln x\,dx$$

$u = \ln x \qquad dv = x\,dx$

$du = \frac{1}{x}dx \qquad v = \frac{1}{2}x^2$

$$\ln y = \frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,dx$$

$$\ln y = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$$

$$\ln 4 = \frac{1}{2}\ln 1 - \frac{1}{4} + c$$

$$\ln 4 + \frac{1}{4} = c$$

$$\ln y = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + \ln 4 + \frac{1}{4}$$

$$\boxed{y = 4e^{\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + \frac{1}{4}}}$$

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (a) and (b) on this page.

Response for question 5(a)

$f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!}$

$T_2 = 4 + 0(x-1) + \frac{4(x-1)^2}{2!}$

$f(2) \approx 4 + 0(2-1) + \frac{4(2-1)^2}{2!}$

$= 4 + \frac{4}{2} = \boxed{4.5}$

Response for question 5(b)

x	y	Δx	$\frac{dy}{dx}$	$\Delta y = \Delta x\left(\frac{dy}{dx}\right)$	$(x+\Delta x, y+\Delta y)$
1	4	.5	0	0	$(1.5, 4)$
1.5	4	.5	$6\ln(1.5)$	$3\ln(1.5)$	$(2, 4+3\ln(1.5))$
2	$4+3\ln(1.5)$				

$f(2) \approx 4 + 3\ln(1.5)$

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 part (c) on this page.

Response for question 5(c)

LIPET

$uv - \int v\,du$ $u = \ln x$ $v = \frac{1}{2}x^2$

$du = \frac{1}{x}dx$ $dv = x\,dx$

$\frac{dy}{dx} = y \cdot (x \ln x)$

$dy = y \cdot (x \ln x)\,dx$

$\int \frac{1}{y}\,dy = \int (x \ln x)\,dx$

$\frac{1}{2}x^2 \ln x - \frac{1}{2}\int x^2 \cdot \frac{1}{x}\,dx$

$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\,dx$

$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$

$\ln|y| = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$ $f(1) = 4$

$\ln 4 = -\frac{1}{4} + C$

$C = \ln 4 + \frac{1}{4}$

$e^{\ln|y|} = e^{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + \ln 4 + \frac{1}{4}}$

$y = e^{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + \underbrace{\ln 4 + \frac{1}{4}}_{C}}$

$y = Ce^{kt} = (\ln 4 + \frac{1}{4})e^{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2}$

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (a) and (b) on this page.

Response for question 5(a)

$f(1) = 4$

$f'(1) = \frac{dy}{dx}\Big|_{(1,4)} = 4(1)\ln 1 = 0$

$f''(1) = \frac{d^2y}{dx^2}\Big|_{(1,4)} \Rightarrow \frac{d}{dx}[yx\ln x]\Big|_{(1,4)}$

$f \approx P_2(x) = 4 + \frac{f''(1)(x-1)^2}{2} = 4 + \frac{\frac{d}{dx}[yx\ln x]\big|_{(1,4)}(x-1)^2}{2}$

$f(2) \approx 4 + \frac{\frac{d}{dx}[yx\ln x]\big|_{(1,4)}(2-1)^2}{2} \rightarrow \boxed{f(2) \approx 4 + \frac{\frac{d}{dx}[yx\ln x]\big|_{(1,4)}}{2}}$

Response for question 5(b)

$f(1) = 4$

$f(1.5) = 4 + [4 \cdot 1(\ln 1)](.5) = 6$

$\boxed{f(2) = 6 + [6 \cdot \tfrac{3}{2}\ln\tfrac{3}{2}](.5)}$

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 part (c) on this page.

Response for question 5(c)

$\frac{dy}{dx} = yx\ln x$

$\int \frac{1}{y}\,dy = \int x\ln x\,dx$

integration by parts

$u = x, \quad dv = \ln x\,dx$

$du = 1\,dx \quad v = \frac{1}{x}$

$\int x\ln x\,dx = x\left(\frac{1}{x}\right) - \int \frac{1}{x}(1)\,dx$

$= 1 - \ln|x| + C$

$\ln|y| = 1 - \ln|x| + C$

↳ find C, $f(1) = 4$

$\ln 4 = 1 - \ln(1) + C$

$C = \ln 4 - 1$

$\ln|y| = 1 - \ln|x| + \ln 4 - 1$

$\ln|y| = \ln 4 - \ln|x|$

$e^{\ln 4 - \ln|x|} = y$

Question 5

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem $y = f(x)$ is the particular solution to $\frac{dy}{dx} = y \cdot (x \ln x)$ with $f(1) = 4$ and students were told that $f''(1) = 4.$ In part (a) students were asked to write the second-degree Taylor polynomial for f about $x = 1$ and to use the polynomial to approximate $f(2)$. A correct response would determine that $f'(1) = 0$ and use this value and the given values of $f(1)$ and $f''(1)$ to write the polynomial $4 + 2(x-1)^2$. The response would then find an approximation of $f(2) \approx 6.$

In part (b) students were asked to use Euler's method to approximate $f(2)$ using two steps of equal size starting at $x = 1.$ A correct response would use Euler's method with $\Delta x = 0.5$ to first approximate $f(1.5) \approx 4$ and then use that value with Euler's method to approximate $f(2) \approx 4 + 3\ln 1.5.$

In part (c) students were asked to find the particular solution $y = f(x)$ with initial condition $f(1) = 4.$ A correct response would use integration by parts to find $\ln|y| = \int x \ln x \, dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C,$ with $C = \ln 4 + \frac{1}{4}$ determined from the initial condition. Then solving for y results in the solution $y = e^{\left(\frac{x^2 \ln x}{2} - \frac{x^2}{4} + \ln 4 + \frac{1}{4}\right)}.$

Sample: 5A
Score: 9

The response earned 9 points: 2 points in part (a), 2 points in part (b), and 5 points in part (c). In part (a) the response earned the first point with a correct expression for the Taylor polynomial in line 2 on the left. Simplification is not necessary. The response earned the second point with a correct approximation of $f(2)$ in line 3 on the left. Again simplification is not necessary. In part (b) the response earned the first point in the table: two Euler steps are visible in the rows, $\Delta x = 0.5$ is given in the second column, the correct values for $\frac{dy}{dx}$ are given in the third column, and the initial condition $f(1) = 4$ is used in the first row. Note that because the final answer is correct in this case, the table does not need to be labeled. The response earned the second point with the correct answer boxed beneath the table. In part (c) the response earned the first point with a correct separation of variables in line 2. The response earned the second point with a correct antiderivative of $x \ln x$ in line 5 on the right. The response earned the third point with a correct antiderivative of $\frac{1}{y}$ in line 4 on the left. No absolute value signs are necessary. The response is eligible for the fourth point. The response earned the fourth point with the correct inclusion of the constant of integration in line 5 and the correct substitution of 1 for x and 4 for y in line 6. The response is eligible for the fifth point, which it earned with the correct answer in line 9.

Question 5 (continued)

Sample: 5B
Score: 7

The response earned 7 points: 1 point in part (a), 2 points in part (b), and 4 points in part (c). In part (a) the response earned the first point with a correct expression for the Taylor polynomial in line 2. Simplification is not necessary. The response did not earn the second point. While the correct approximation for $f(2)$ is shown in line 3 and in line 4 on the left, the boxed result is simplified incorrectly. In part (b) the response earned the first point: two Euler steps are seen in the second and third rows of the table, the step size $\Delta x = 0.5$ is shown in the third column of the table, the correct expression for $\frac{dy}{dx}$ is used to create the fourth column of the table, and the initial condition $f(1) = 4$ is shown in the first two entries of the second row of the table. Note that because the answer is correct in this case, the table does not need to be labeled. The response earned the second point with a correct answer below the table. In part (c) the response earned the first point with a correct separation of variables in line 3 on the left side of the page. The response earned the second point with a correct antiderivative of $x \ln x$ in lines 1-5 on the right side of the page. The response earned the third point with a correct antiderivative of $\frac{1}{y}$ in line 4 on the left side of the page. The response is eligible for the fourth point as it has earned the first point and at least one of the antiderivative points. The response earned the fourth point with a correct inclusion of the constant of integration in line 3 on the left side of the page and the substitution of 1 for x and 4 for y in line 4 on the left side of the page. The response did not earn the fifth point as the answer in line 9 on the left side of the page is incorrect. Note that the answer given in line 8 of the left side of the page is correct, but is simplified incorrectly.

Sample: 5C
Score: 4

The response earned 4 points: no points in part (a), 1 point in part (b), and 3 points in part (c). In part (a) the response did not earn the first point because no correct Taylor polynomial is presented. The response did not earn the second point because no correct approximation for $f(2)$ is presented. In part (b) the response earned the first point: two Euler steps are shown in lines 2 and 3, the step size $\Delta x = 0.5$ is used in lines 2 and 3, the correct expression for the derivative is evaluated in lines 2 and 3, and the initial condition $f(1) = 4$ is stated in line 1. Note that the response contains a single error: the value of the approximation for $f(1.5)$ in line 2 is simplified incorrectly. Importing this incorrect value correctly into line 3 is not an error. Because only one mistake is made, the response is still eligible for the first point, which it earned. The response did not earn the second point because the boxed answer is incorrect. In part (c) the response earned the first point with a correct separation of variables in line 2 on the left side of the page. The response did not earn the second point as the antiderivative for $x \ln x$ is incorrect. The response earned the third point with a correct antiderivative for $\frac{1}{y}$ in line 3 on the left side of the page. The response is eligible for the fourth point because it earned the first point and one of the two antiderivative points. The response earned the fourth point with a correct inclusion of a constant of integration in line 3 on the left side of the page and a correct substitution of 1 for x and 4 for y in line 5 on the left side of the page. The response is not eligible for the fifth point as it did not earn all of the first four points.

2021

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AP® Calculus BC

Sample Student Responses and Scoring Commentary

Inside:

Free Response Question 6

- ☑ Scoring Guideline
- ☑ Student Samples
- ☑ Scoring Commentary

Part B (BC): Graphing calculator not allowed
Question 6 — 9 points

General Scoring Notes

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.

The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

Model Solution	Scoring	

(a) State the conditions necessary to use the integral test to determine convergence of the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$. Use the integral test to show that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.

e^{-x} is positive, decreasing, and continuous on the interval $[0, \infty)$.	Conditions	**1 point**
To use the integral test to show that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges, show that $\int_0^{\infty} e^{-x}\,dx$ is finite (converges).	Improper integral	**1 point**
$\int_0^{\infty} e^{-x}\,dx = \lim_{b\to\infty} \int_0^b e^{-x}\,dx = \lim_{b\to\infty} \left(-e^{-x}\Big\vert_0^b\right) = \lim_{b\to\infty}\left(-e^{-b} + e^0\right) = 1$ Because the integral $\int_0^{\infty} e^{-x}\,dx$ converges, the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.	Evaluation	**1 point**

Scoring notes:

- To earn the first point a response must list all three conditions: e^{-x} is positive, decreasing, and continuous.
- The second point is earned for correctly writing the improper integral or for presenting a correct limit equivalent to the improper integral (for example, $\lim_{b \to \infty} \int_0^b e^{-x}\, dx$).
- To earn the third point a response must correctly use limit notation to evaluate the improper integral, find an evaluation of e^0 (or 1), and conclude that the integral converges or that the series converges.
- If an incorrect lower limit of 1 is used in the improper integral, then the second point is not earned. In this case, if the correct limit ($1/e$) is presented, then the response is eligible for the third point.
- If the response only relies on using a geometric series approach, then no points are earned [0-0-0].
- A response that presents an evaluation with ∞, such as $e^{-\infty} = 0$, does not earn the third point.

Total for part (a) 3 points

(b) Use the limit comparison test with the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ to show that the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.

$$\lim_{n\to\infty} \frac{\frac{1}{e^n}}{\left\lvert \frac{(-1)^n}{2e^n + 3} \right\rvert} = \lim_{n\to\infty} \frac{2e^n + 3}{e^n} = 2$$ The limit exists and is positive. Therefore, because the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges, the series $\sum_{n=0}^{\infty} \left\lvert \frac{(-1)^n}{2e^n + 3} \right\rvert$ converges by the limit comparison test. Thus, the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.	Sets up limit comparison	**1 point**
	Explanation	**1 point**

Scoring notes:

- The first point is earned for setting up the limit comparison, with or without absolute values. Limit notation is required to earn this point.
- The reciprocal of the given ratio is an acceptable alternative; the limit in this case is $1/2$.
- The second point cannot be earned without the use of absolute value symbols, which can occur explicitly or implicitly (e.g., a response might set up the limit comparison initially as $\lim_{n\to\infty} \frac{\frac{1}{e^n}}{\frac{1}{2e^n + 3}}$).
- Earning the second point requires correctly evaluating the limit and noting that the limit is a positive number. For example, $L = 2 > 0$ or $L = 1/2 > 0$. Therefore, comparing the limit L to 1 does not earn the explanation point.
- A response does not have to repeat that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.
- A response that draws a conclusion based only on the sequence (such as $\frac{1}{e^n}$) without referencing a series does not earn the second point.
- If the response does not explicitly use the limit comparison test, then no points are earned in this part.
- A response cannot earn the second point for just concluding that "the series" converges absolutely because there are multiple series in this part of the problem. The response must specify that the series $g(1)$ or $\sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.

Total for part (b) 2 points

(c) Determine the radius of convergence of the Maclaurin series for g.

$\left\lvert \frac{(-1)^{n+1} x^{n+1}}{2e^{n+1}+3} \cdot \frac{2e^n+3}{(-1)^n x^n} \right\rvert = \left\lvert \frac{(2e^n+3)x^{n+1}}{(2e^{n+1}+3)x^n} \right\rvert = \frac{2e^n+3}{2e^{n+1}+3}\lvert x\rvert$	Sets up ratio	**1 point**
$\lim_{n\to\infty} \frac{2e^n+3}{2e^{n+1}+3}\lvert x\rvert = \frac{1}{e}\lvert x\rvert$	Computes limit of ratio	**1 point**
$\frac{1}{e}\lvert x\rvert < 1 \Rightarrow \lvert x\rvert < e$ The radius of convergence is $R = e$.	Answer	**1 point**

Scoring notes:

- The first point is earned for $\frac{(-1)^{n+1} x^{n+1}}{2e^{n+1}+3} \cdot \frac{2e^n+3}{(-1)^n x^n}$ or the equivalent. Once earned, this point cannot be lost.
- The second point cannot be earned without the first point.
- To be eligible for the third point the response must have found a limit for a presented ratio such that the limiting value of the coefficient on $\lvert x\rvert$ is finite and not 0. The third point is earned for setting up an inequality such that the limit is less than 1, solving for $\lvert x\rvert$, and interpreting the result to find the radius of convergence.
- The radius of convergence must be explicitly presented, for example, $R = e$. The third point cannot be earned by presenting an interval, for example $-e < x < e$, with no identification of the radius of convergence.

Total for part (c) 3 points

(d) The first two terms of the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ are used to approximate $g(1)$. Use the alternating series error bound to determine an upper bound on the error of the approximation.

The terms of the alternating series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ decrease in magnitude to 0. The alternating series error bound for the error of the approximation is the absolute value of the third term of the series. $\text{Error} \le \left\lvert \frac{(-1)^2}{2e^2 + 3} \right\rvert = \frac{1}{2e^2 + 3}$	Answer	**1 point**

Scoring notes:

- A response of $\frac{1}{2e^2 + 3}$ earns this point.

Total for part (d) 1 point

Total for question 6 9 points

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)

The function $\frac{1}{e^n}$ is continuous, positive, and decreasing for $n \geq 0$

$\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges by integral test

$= \lim_{b \to \infty} \int_0^b \frac{1}{e^n}\, dn = \lim_{b \to \infty} \int_0^b e^{-n}\, dn$

$= \lim_{b \to \infty} \left[-e^{-n}\right]_0^b = \lim_{b \to \infty} \left[-e^{-b} - (-e^0)\right]$

$= 0 + 1 = 1$

Response for question 6(b)

$g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$

compare to $\sum_{n=0}^{\infty} \frac{1}{e^n}$ geometric $r = \frac{1}{e} < 1$ converges

absolute value of $g(1)$ ↘

$\lim_{n \to \infty} \frac{\frac{1}{2e^n + 3}}{\frac{1}{e^n}} = \lim_{n \to \infty} \frac{e^n}{2e^n + 3} = \frac{1}{2} > 0$

$\frac{1}{2e^n + 3} > 0$

$\frac{1}{e^n} > 0$

$g(1)$ converges absolutely by the Limit Comparison Test

Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n+3}$$

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{2e^{n+1}+3} \cdot \frac{2e^n+3}{x^n} \right| = \left| \frac{x}{e} \right|$$

$$\left| \frac{x}{e} \right| < 1$$

$$|x| < e$$

$$R = e$$

↑ radius of convergence for g

Response for question 6(d)

approximation using first two terms

Error ≤ | third term of the series g(1) |

$$\text{Error} \le \left| \frac{1}{2e^2+3} \right|$$

$$\text{Error} \le \frac{1}{2e^2+3}$$

Sample 6B

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (a) and (b) on this page.

$e^{-x} = -e^{-x}$

Response for question 6(a)

$\sum_{n=0}^{\infty} \frac{1}{e^n} = \frac{1}{e^x}$ *function is equal to series
*series can be approximated by function

$\int_0^{\infty} \frac{1}{e^x} = \lim_{b \to \infty} \int_0^b \frac{1}{e^x} = \lim_{b \to \infty} -\frac{1}{e^x}\Big|_0^b = \lim_{b \to \infty} -\frac{1}{e^b} + \frac{1}{e^0} = 1$

Since $\int_0^{\infty} \frac{1}{e^x} = 1$, $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges by the integral test.

Response for question 6(b)

$\sum_{n=0}^{\infty} \frac{1}{e^n} > \sum_{n=0}^{\infty} \frac{1}{2e^n+3}$

$\lim_{n \to \infty} \left| \frac{e^n}{2e^n+3} \right| = \lim_{n \to \infty} \left| \frac{e^n}{2e^n} \right| = \frac{1}{2} < 1$

So, converges absolutely by limit comparison test.

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\cdot x^{n+1}}{(-1)^n\cdot x^n}\cdot\frac{2e^n+3}{2e^{n+1}+3}\right| = \lim_{n\to\infty}\left|(-1)(x)\cdot\frac{1}{2e}\right|$$

$$-1 < -\frac{x}{2e} < 1$$

$$2e > x > -2e$$

radius of convergence $= 2e$

Response for question 6(d)

*since first two terms are used ($n=0$ and $n=1$) for approximation, third term will give the upper bound on error of approximation

$$\text{error} \le \frac{(-1)^2}{2e^2+3} \rightarrow n=2$$

$$\text{error} \le \frac{1}{2e^2+3}$$

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)

Conditions are positive, continuous, and decreasing.

$\int_0^\infty e^{-n}\,dn = -e^{-n}\Big|_0^\infty = \frac{-1}{e^n}\Big|_0^\infty = [0 - \frac{-1}{e^0}] = \frac{1}{1} = 1$

Since the integral went to a finite value in 1, the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges. (The integral test does not say where it converges to though)

Response for question 6(b)

~~$\lim_{n\to\infty} \frac{\frac{1}{e^n}}{\frac{(-1)^n}{2e^n+3}} = \lim_{n\to\infty} \frac{2e^n+3}{(-1)^n e^n} = \lim_{n\to\infty} \frac{2e^n+3}{e^n} = 2$~~

$\lim_{n\to\infty} \frac{\frac{(-1)^n}{2e^n+3}}{\frac{1}{e^n}} = \lim_{n\to\infty} \frac{(-1)^n e^n}{2e^n+3} \rightarrow \lim_{n\to\infty} \left|\frac{e^n}{2e^n+3}\right| = \frac{1}{2}$

The limit test converges to less than 1, so it converges there absolutely and thus there is no need to check conditional convergence.

Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{2e^{n+1}+3}\cdot\frac{2e^{n}+3}{(-1)^{n}x^{n}}\right| = \lim_{n\to\infty}\left|\frac{(-1)x+3}{2e+3}\right| \to \frac{-x+3}{2e+3} < 1$$

$-x+3 < 2e+3$

$-x < 2e$

$x > 2e$

$3 < x < 2e+3$

Response for question 6(d)

need 3rd term

$\frac{-1}{2e^3+3}$

Upper bound on error of the approximation can only be as big as the next term, thus the maximum error is $-\frac{1}{2e^3+3}$

Question 6

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem the function g has derivatives of all orders for all real numbers and students were given the Maclaurin series for g.

In part (a) students were asked to state the conditions necessary to use the integral test to determine whether the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges, and then to use the integral test to show that the series converges. A correct response should state that the integral test requires the function $\frac{1}{e^x}$ to be positive, decreasing, and continuous on the interval $[0, \infty)$. The response should continue by demonstrating that the improper integral $\int_0^{\infty} e^{-x}\, dx$ is finite and therefore converges, so $\sum_{n=0}^{\infty} e^{-n}$ also converges.

In part (b) students were asked to use the limit comparison test with the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ to show that the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely. A correct response should use correct notation to show that $\lim_{n\to\infty} \frac{\frac{1}{e^n}}{\left|\frac{(-1)^n}{2e^n + 3}\right|}$ is finite and positive and reference the convergence of $\sum_{n=0}^{\infty} \frac{1}{e^n}$ determined in part (a).

In part (c) students were asked to determine the radius of convergence of the Maclaurin series for g. A correct response should use the ratio test to determine the radius of convergence is $R = e$.

In part (d) students were asked to use the alternating series error bound to determine an upper bound on the error when the first two terms of the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ are used to approximate $g(1)$. A correct response should indicate that the approximate error is bounded by the absolute value of the third term of the series, $\frac{1}{2e^2 + 3}$.

Sample: 6A
Score: 9

The response earned 9 points: 3 points in part (a), 2 points in part (b), 3 points in part (c), and 1 point in part (d). In part (a) the response earned the first point in the first line by stating that "the function $\frac{1}{e^n}$ is continuous, positive, and decreasing." The response earned the second point in the fourth line by presenting the limit $\lim_{b\to\infty} \int_0^b \frac{1}{e^n}\, dn$. The response earned the third point by presenting a correct antiderivative on the fifth line, correctly evaluating the limit on the sixth line, and stating a correct conclusion on the third line. In part (b) the response earned the first point

Question 6 (continued)

in the second line by presenting the limit (with or without absolute values) $\lim_{n\to\infty} \frac{\frac{1}{2e^n+3}}{\frac{1}{e^n}}$. The response earned the second point by correctly evaluating the limit resulting in $\frac{1}{2}$, referencing that $\frac{1}{2} > 0$, and presenting a correct conclusion in the lower right. In part (c) the response earned the first point in the second line by presenting the ratio (with or without a limit) $\left|\frac{x^{n+1}}{2e^{n+1}+3} \cdot \frac{2e^n+3}{x^n}\right|$. In the absence of the factors $(-1)^{n+1}$ and $(-1)^n$, the response must have absolute values to earn this point. The response earned the second point on the second line by correctly evaluating the limit. The response earned the third point on the fifth line by correctly identifying the radius of convergence. In part (d) the response earned the point by providing the correct answer.

Sample: 6B
Score: 6

The response earned 6 points: 2 points in part (a), 1 point in part (b), 2 points in part (c), and 1 point in part (d). In part (a) the response did not earn the first point because the response does not identify the three needed conditions for the function $\frac{1}{e^x}$. The response earned the second point in the second line by presenting the improper integral (with or without a differential) $\int_0^\infty \frac{1}{e^x}$. The response earned the third point by presenting a correct antiderivative, correctly evaluating the limit, and stating a correct conclusion. In part (b) the response earned the first point in the second line by presenting the limit (with or without absolute values) $\lim_{n\to\infty} \left|\frac{e^n}{2e^n+3}\right|$. The response did not earn the second point because the response compares $\frac{1}{2}$ with 1 and not 0. In part (c) the response earned the first point in the first line by presenting the ratio (with or without a limit or absolute values) $\frac{(-1)^{n+1} x^{n+1}}{(-1)^n x^n} \cdot \frac{2e^n+3}{2e^{n+1}+3}$. The response did not earn the second point because the response, by bounding $\frac{-x}{2e}$ between both -1 and 1, implies that their limit is $\frac{-x}{2e}$, which is incorrect. The response is eligible for the third point because the response presents a value for the limit and considers an absolute value by presenting $-1 < \frac{-x}{2e} < 1$. The response earned the third point by presenting the radius of convergence $2e$, which is consistent with their limit. In part (d) the response earned the point by providing the correct answer.

Question 6 (continued)

Sample: 6C
Score: 3

The response earned 3 points: 1 point in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d). In part (a) the response did not earn the first point because the response states the correct conditions but does not reference the function $\frac{1}{e^x}$. The response earned the second point in the second line by presenting the improper integral $\int_0^\infty e^{-n}\, dn$. The response did not earn the third point because on the second line, the response applies the Fundamental Theorem of Calculus to an improper integral. In part (b) the response earned the first point on the first line by presenting the limit (with or without absolute values) $\lim_{n\to\infty} \frac{\frac{(-1)^n}{2e^n + 3}}{\frac{1}{e^n}}$. The response did not earn the second point because in the second line, the response references a limit "less than 1" whereas the correct application of the limit comparison test would require referencing a limit greater than 0. In part (c) the response earned the first point on the first line by presenting the ratio (with or without a limit or absolute values) $\frac{(-1)^{n+1} x^{n+1}}{2e^{n+1} + 3} \cdot \frac{2e^n + 3}{(-1)^n x^n}$. The response did not earn the second point because the evaluation of the limit is incorrect. The response did not earn the third point because the response does not identify a radius of convergence consistent with their limit evaluation. In part (d) the response did not earn the point as the response presents the incorrect answer.